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General Asymptotic Formulation for the Bifurcation Problem of Thin Walled Structures in Contact with Rigid Surfaces

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The bifurcation problem of thin walled structures in contact with rigid surfaces is formulated by adopting the multiple scales asymptotic technique. The general theory developed in this paper is very useful for the bifurcation analysis of waviness instabilities in the sheet metal forming. The formulation is presented in a full Lagrangian formulation. Through this general formulation, the bifurcation functional is derived within an error of $O(E^4)$ (E : shell's thickness parameter). This functional can be used in numerical solutions to sheet metal forming instability problem.

Key Words : Bifurcation Functional, Thin Walled Structures, Rigid Surfaces, Multiple Scales Asymptotic Technique, Full Lagrangian Formulation

1. Introduction and Motivation

The present work is motivated by the desire to provide a consistent and general method for modelling bifurcation instabilities in thin walled structures, especially the case of shells in contact with a rigid punch and constitutes in essence the extended work of an earlier study by Triantafyllidis and Kwon (1987) and Kwon (1992).

Besides the theoretical importance of the aforementioned task there are some significant practical aspects to it as well. More specifically of particular concern to this work is the modelling of surface waviness instabilities associated with sheet metal forming problems (Kim et al., 1999). These type instabilities (due to the presence of longitudinal compressive stresses in parts of the sheet) depend strongly on the sheet's geometry and boundary conditions and cannot be modelled using simple local (Forming Limit Diagram type) criteria. The only way to solve these problems was by employing the classical approach developed essentially by Koiter (1945) and which is based on the use of the two dimensional nonlinear shell theory.

When this classical approach is employed, for instabilities deep in the plastic range, the results depend strongly on the nonlinear shell theory used. In addition, the numerical (usually finite element) membrane theory codes that have been developed for the modelling of localization type instabilities of the sheet are completely useless for they lack bending stiffness, the essential mechanism for the generation of the surface type instabilities in question.

It is the purpose of this investigation to propose a consistent as well as general alternative method for the calculation of critical loads and eigenmodes in a shell buckling problem in a unique way, once the shell's geometry and constitutive equations have been determined. The proposed general method also provides the critical loads and eigenmodes as a function of the shell's thickness to any desired degree of accuracy, information which is impossible to obtain using the

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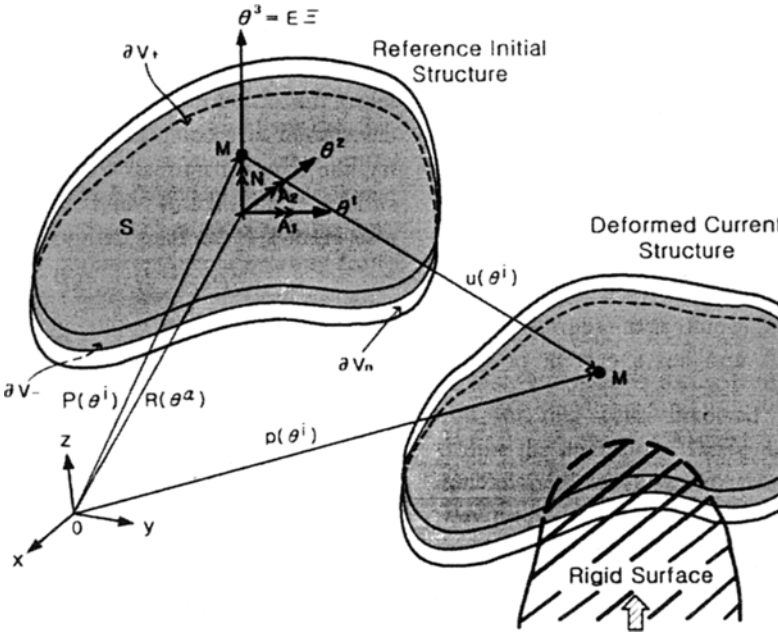

Fig. 1 Schematic diagram of thin walled shell structure in contact with rigid surface and sign conventions (the shadow surface S denotes the middle surface of the shell structure, and M denotes the material point)

aforedescribed two dimensional nonlinear shell theory approach.

In place of the classical method, in which a two dimensional nonlinear shell theory (derived from the three dimensional governing equations of the solid) is linearized about the critical load, the present approach departs from the bifurcation instability equations of the three dimensional shell like solid (which have been obtained by linearization about the critical load of the same three dimensional governing equations for the nonlinear solid in question) and subsequently takes the limit as the shell thickness tends to zero, following a multiple scales asymptotic technique in the spirit of the approach introduced by Destuynder (1980) for the derivation of linearly elastic shell theories. This novel method together with an analytically tractable application has first been presented by Triantafyllidis and Kwon (1987). Here the proposed methodology is generalized in order to provide the asymptotic expansions for the critical load and mode for a shell like solid with arbitrary middle surface geometry, and valid for a wide variety of constitutive laws (any rate independent material is encompassed by the analysis). A full Lagrangian formulation for the determination of the minimum value of the solid's bifurcation functional is given and an explanation of the usefulness of these results for numerical computations in the stability of sheet metal forming problems is also provided.

The presentation is concluded with some comments on the significance of the presented results as well as a discussion for further work in this direction.

2. Kinematical and Constitutive Preliminaries; Three Dimensional Formulation of the Bifurcation Problem

Consider a shell-like solid which in its reference configuration occupies a volume V with the boundary $\partial V = \partial V_+ \cup \partial V_n \cup \partial V_-$ as depicted in Fig. 1, where ∂V_+, ∂V_n, ∂V_- are the top, lateral and bottom surfaces of the solid respectively. For convenience, the following parametrization for the solid is adopted. Let $\boldsymbol{R}(\theta^\alpha)$ be the position vector of an arbitrary point on the reference configuration's middle surface (denoted by S). Then the position vector $\boldsymbol{P}(\theta^i)$ of an arbitrary material point M of the shell is given by

$$\boldsymbol{P}(\theta^i)=\boldsymbol{R}(\theta^\alpha)+\theta^3\boldsymbol{N}(\theta^\alpha),\quad -H\le\theta^3\le H \quad (1)$$

where $\boldsymbol{N}(\theta^\alpha)$ is the unit outward normal vector to S at $\boldsymbol{R}(\theta^\alpha)$ and θ^3 the distance of point M from the shell's middle surface S. In addition, 2H denotes the thickness of the shell at position $\boldsymbol{R}$.

Under the action of a monotonically increasing load λ, a material point with coordinates θ^i displaces by $\boldsymbol{u}(\theta^i)$ and has a current position vector $\boldsymbol{p}(\theta^i)$ with

$$\boldsymbol{p}(\theta^i)=\boldsymbol{P}(\theta^i)+\boldsymbol{u}(\theta^i) \quad (2)$$

The covariant basis vectors $\boldsymbol{G}_i$ in the reference and $\boldsymbol{g}_i$ in the current configuration respectively are found in terms of the corresponding position vectors as

$$\boldsymbol{G}_i=\frac{\partial \boldsymbol{P}}{\partial\theta^i},\ \boldsymbol{g}_i=\frac{\partial \boldsymbol{p}}{\partial\theta^i} \quad (3)$$

and they are related to each other by the following relation

$$\boldsymbol{g}_i=F_{ki}\ \boldsymbol{G}^k,\ F_{ki}=u_{k|i}+G_{ki} \quad (4)$$

where a vertical stroke followed by an index denotes covariant differentiation with respect to the corresponding coordinate in the reference configuration.

At this point, the relations between the reference covariant basis $\boldsymbol{G}_i$ and the reference midsurface covariant basis $\boldsymbol{A}_i$ will be introduced. The midsurface basis $\boldsymbol{A}_i$ is defined by

$$\boldsymbol{A}_\alpha=\frac{\partial \boldsymbol{R}}{\partial\theta^\alpha}=\boldsymbol{G}_\alpha(\theta^1,\theta^2,0),\ \boldsymbol{A}_3=\boldsymbol{N} \quad (5)$$

and hence from Eq. (3) using also Eq. (1) and Eq. (5) one obtains

$$\boldsymbol{G}_i=M_i^j\boldsymbol{A}_j\ ;\ M_\alpha^\beta=\delta_\alpha^\beta-\theta^3B_\alpha^\beta,\ M_\alpha^3=M_3^\beta=0,\ M_3^3=1 \quad (6)$$

where B_α^β is the curvature tensor for the reference middle surface. Also of interest are the expressions for the derivatives of $\boldsymbol{A}_i$ with respect to θ^j which are found to be

$$\frac{\partial \boldsymbol{A}_i}{\partial\theta^j}=T_{ij}^k\ \boldsymbol{A}_k\ ;\ T_{\alpha\beta}^\gamma=\Gamma_{\alpha\beta}^\gamma,\ T_{\alpha\beta}^3=B_{\alpha\beta},\ T_{3\alpha}^\beta=-B_\beta^\alpha \quad (7)$$

with all other components of T_{jk}^i being zero. In Eq. (7) $\Gamma_{\alpha\beta}^\gamma$ and B_α^β are respectively the Christoffel symbols (Kwon, 1992) and curvature tensor of the shell's reference middle surface. It is interesting to note that in Eq. (7) the symbols T_{jk}^i are nothing else but the three dimensional Christoffels of the second kind evaluated at $\theta^3=0$. For the description of a completely general shell geometry, the outward normals to the top and bottom surface of the shell $\boldsymbol{N}^+$ and $\boldsymbol{N}^-$ respectively are also needed. From their definitions

$$\boldsymbol{N}^+=\left(\frac{\partial \boldsymbol{P}(\theta^\alpha,H)}{\partial\theta^1}\times\frac{\partial \boldsymbol{P}(\theta^\alpha,H)}{\partial\theta^2}\right)/\left\|\frac{\partial \boldsymbol{P}(\theta^\alpha,H)}{\partial\theta^1}\times\frac{\partial \boldsymbol{P}(\theta^\alpha,H)}{\partial\theta^2}\right\| \quad (8)$$

$$\boldsymbol{N}^-=\left(\frac{\partial \boldsymbol{P}(\theta^\alpha,-H)}{\partial\theta^1}\times\frac{\partial \boldsymbol{P}(\theta^\alpha,-H)}{\partial\theta^2}\right)/\left\|\frac{\partial \boldsymbol{P}(\theta^\alpha,-H)}{\partial\theta^1}\times\frac{\partial \boldsymbol{P}(\theta^\alpha,-H)}{\partial\theta^2}\right\|$$

and with the help of Eq. (1) and Eq. (5), one obtains

$$\boldsymbol{N}^+=N_i^+\ \boldsymbol{G}^i\ ;\ N_i^+=(-\delta_i^3-H_{,\alpha}\ \delta_i^\alpha)/(1+H_{,\gamma}\ H_{,\delta}\ G^{\gamma\delta}) \quad (9)$$

$$\boldsymbol{N}^-=N_i^-\ \boldsymbol{G}^i\ ;\ N_i^-=(-\delta_i^3-H_{,\alpha}\ \delta_i^\alpha)/(1+H_{,\gamma}\ H_{,\delta}\ G^{\gamma\delta})$$

where we note that here and subsequently $(\ )_{,\alpha}$ denotes partial differentiation with respect to θ^α, $\boldsymbol{G}^i$ is the contravariant reference basis, and $G^{ij}=\boldsymbol{G}^i\cdot\boldsymbol{G}^j$ is the contravariant metric. The results in Eq. (9) hold for the general case of a variable thickness shell, i. e., $H=H(\theta^\alpha)$. In the event that the reference configuration of the shell has a uniform thickness 2H, Eq. (9) simplifies to the obvious result $\boldsymbol{N}^+=\boldsymbol{A}_3=-\boldsymbol{N}^-$.

Attention is next focused on the constitutive equations. Of interest in all stability analyses for rate independent solids (hyperelastic, hypoelastic or more generally elastic-plastic) is the rate form of their constitutive equation, which most often is given in the form of the fourth order incremental moduli tensor $\mathcal{L}^{ijkl}$ relating the convective rate of the Kirchhoff stress $\overset{\triangledown}{\tau}{}^{ij}$ to the strain rate tensor D_{kl} in the current configuration (or equivalently relating the rate of the Second Piola-Kirchhoff stress $\dot{S}^{ij}$ to the rate of the Lagrange-Green strain $\dot{E}_{kl}$ in the reference configuration), namely

$$\overset{\triangledown}{\tau}{}^{ij}=\mathcal{L}^{ijkl}D_{kl}\quad(\dot{S}^{ij}=\mathcal{L}^{ijkl}\ \dot{E}_{kl}) \quad (10)$$

For most rate independent materials encountered in the application the incremental moduli tensor has the following symmetries

$$\mathscr{L}^{ijkl}=\mathscr{L}^{klij}=\mathscr{L}^{jikl}=\mathscr{L}^{ijlk} \quad (11)$$

It should be noted at this point that the first one in Eq. (11) is very important, for it insures the existence of a bifurcation functional, say , quadratic in terms of the bifurcation eigenmode v_i, which loses its positive definiteness for the first time, as the load parameter increases away from zero, at the onset of the first bifurcation instability. Following Hill (1957, 1958) and Hutchinson (1974), the bifurcation functional is given by

$$\mathrm{F}[\lambda, v]=\frac{1}{2}\int_V L^{ijkl}v_{i|j}v_{k|l}dV ;$$
$$L^{ijkl}=\mathscr{L}^{sjql}F_{rs}F_{pq}G^{ri}G^{pk}+\tau^{lj}G^{ki} \quad (12)$$

where the covariant differentiation is with respect to the reference metric and F_{ij} is the deformation gradient tensor. At the onset of the first bifurcation occurring for $\lambda=\lambda_{cr}$, the first variation of F, $\delta\mathrm{F}=0$. The functional form of F in Eq. (12) is valid for the case of dead surface and body loads. For the case of configuration dependent loading, i. e., contact of the shell with an obstacle, some additional surface terms will be required for F as it will be explained in the appropriate sections.

The minimum eigenvalue of F, say B, is defined by

$$B(\lambda)=\min[2\mathrm{F}[\lambda, v]/\|v\|^2] ; \|v\|^2$$
$$=\int_V G^{ij}v_iv_jdV \quad (13)$$

and for a given shell geometry $B(\lambda)>0$ for $\lambda<\lambda_{cr}$ while $B(\lambda)=0$ for $\lambda=\lambda_{cr}$ the lowest critical load for F. The eigenvalue B and the corresponding eigenfunction v for Eq. (13) are found from the variational equation $\delta\mathrm{F}=B\delta(\|v\|^2/2)$.

For reasons of consistency, throughout the rest of this presentation tensor components are usually considered with respect to the reference midsurface basis $\boldsymbol{A}_i$ and are denoted by a bar $(\bar{\ })$ surmounting the component in question. The absence of a bar in a tensor component implies the use of the three dimensional reference basis $\boldsymbol{G}_i$.

Hence from Eq. (12) the components of the incremental moduli tensor $\mathbf{L}$ with respect to $\boldsymbol{A}_i$ are related to their counterparts with respect to $\boldsymbol{G}_i$ by

$$L^{ijkl}=\bar{L}^{pqrs}Q^i_pQ^j_qQ^k_rQ^l_s ; \ Q^i_j=(M^i_j)^{-1} \quad (14)$$

while the corresponding relation for the components of the bifurcation velocity gradient tensor $v\nabla$ are

$$v_{i|j}=(\partial\bar{v}_k/\partial\theta^j-T^n_{kj}\bar{v}_n)M^k_i \quad (15)$$

where M^i_j is given by Eq. (6). It is interesting to note that from Eq. (6) easily follows

$$Q^\alpha_\beta=(M^\alpha_\beta)^{-1}, \ Q^3_\beta=Q^\alpha_3=0, \ Q^3_3=1 \quad (16)$$

Next, attention is focused on the proper selection for the scales involved in this problem. Unlike the analysis of Destuynder(1980) for the derivation of the linear elastic shell approximations, only one small parameter is required in this problem. In the asymptotic analysis considered, the small quantity is identified with $\mathrm{E}=\max H(\theta_\alpha)$. The shell's reference middle surface is considered fixed during the limiting process of E tending to zero. Thus E is the only small parameter pertinent to this problem and the corresponding rescaled variable is defined by

$$\Xi=\theta^3/E, \ -Z(\theta^\alpha)\leq\Xi\leq Z(\theta^\alpha) \quad (17)$$

with the obvious definition $Z(\theta^\alpha)=H(\theta^\alpha)/E$. In the most common case of a shell with uniform reference configuration thickness $Z=$ const.. More generally, $Z=O(1)$ while $Z,_\alpha=O(E^p)$, $p\in N$, depending on the nonuniformities in the reference configuration.

3. Asymptotic Bifurcation Problem Formulation for Shells in Contact with Rigid Surfaces

An application of the general asymptotic shell stability theory, which is of great importance to sheet metal forming process, is the case of a shell in contact with a rigid surface. In cold forming processes of metalic sheets a rigid punch advances against a thin (relatively to the punch characteristic dimensions) sheet, held in place between a blank-holder and a die. Due to the peculiarities of the punch and die geometry, longitudinal compressive stresses do develop in parts of the sheet. If the compressive stresses in the unsupported part of the sheet become adequately high, surface waviness type instabilities will occur in that part of the shell which in the literature carry the name of wrinkling or puckering and see Devons (1941) for more details.

As explained in the introduction, the classical modelling of this phenomenon available up to date, requires a finite strain nonlinear shell model for the sheet with the associated disadvantages of inconsistency for such a choice, in addition, and equally important from a practical standpoint, the reliable and easily available membrane theory models of the sheet punching process are useless to the classical approach.

In this section it will be shown that the onset of a surface waviness instability can be found from the loss of positive definiteness of the shell's stability functional which can be computed to any degree of accuracy (in terms of the shell's thickness) required. For the application of practical interest in sheet forming an $O(E^4)$ accuracy in the functional is adequate. Moreover the coresponding calculations require only the knowledge of the membrane prebuckling solution.

Given that in the majority of sheet metal forming applications, an analytical solution is impossible, a full Lagrangian approach for the problem will be adopted in view of its suitability to numerical calculations (since all field quantities are referred to the known initial configuration).

The starting point for the analysis is again the variational statement of the fact that $B(\lambda, E)$ is the minimum eigenvalue for the bifurcation functional Eq. (13), properly modified in order to take into account the contact between the sheet and the punch. More specifically, assuming that due to the contact, the system's potential energy is augmented by

$$\textit{Contact Energy}: \frac{1}{2}\int_{\partial V} k' d^2 \mathrm{H}(d)\, ds \qquad (18)$$

where $d(\lambda, \theta^i)$ is the distance between a point on the surface of the shell with coordinates θ^i and the surface of the punch, k' is the foundation stiffness for the punch (an adequately large value of k' will ensure a very small value of d, so that the penetration distance of the shell on the punch is negligible) and $H(d)$ is the Heaviside function (with $H(d)=1$ for $d \geq 0$, $H(d)=0$ for $d<0$). It is understood that the penetration distance d is signed with $d<0$ for points outside the punch's surface, $d=0$ for points on the surface and $d>0$ for points penetrating the punch.

Hence, in the case of the presence of a punch, the bifurcation functional is augmented by the second Frechet derivative of Eq. (18), an assumption justified by the fact that the bifurcation functional is the second Frechet derivative of the structure's potential energy (assuming of course that such an energy can be constructed). It should also be noted at this point that the contribution of friction into the stability functional has been ignored. Since in most of the applications considered, the bifurcation mode amplitude is expected to be maximized in the unsupported parts of the sheet and to be negligible in the contact areas, such an assumption seems well justified. Note that friction neglection concerns only the buckling mode and not the prebifurcation solution which can (and usually does) strongly depend on friction between punch and sheet (Oden et al., 1983; Kwak, 1990).

Denoting by s^α the curvilinear coordinates in the parametrization of the rigid surface, $\boldsymbol{p}^- \equiv \boldsymbol{p}(\theta^\alpha, -EZ)$ is the position vector of a point in the sheet's lower surface (only the lower surface of the sheet is supposed to be in contact with the punch for the sake of simplicity) and $\boldsymbol{p}^s$ is the position vector of the foot of the normal from $\boldsymbol{p}^-$ to the punch surface, and noting that

$$d^2 = (\boldsymbol{p}^- - \boldsymbol{p}^s) \cdot (\boldsymbol{p}^- - \boldsymbol{p}^s) \qquad (19)$$

the contact term in the bifurcation functional is found to be (assuming that the punch surface is not flat)

$$\int_{\partial V_-} k' \bar{D}^{ij} \bar{v}_i \bar{v}_j dS \qquad (20)$$

with $$\bar{D}^{ij}\bar{v}_i\bar{v}_j = \mathrm{H}(d)\left[\boldsymbol{v} \cdot \boldsymbol{v} + \left(\frac{\partial \boldsymbol{p}^s}{\partial s^\alpha} \cdot \boldsymbol{v}\right) \left(\frac{\partial \boldsymbol{p}^s}{\partial s^\beta} \cdot \boldsymbol{v}\right) \Phi_{\alpha\beta}{}^{-1} \right]$$

$$\Phi_{\alpha\beta} \equiv (\boldsymbol{p}^- - \boldsymbol{p}^s) \cdot \frac{\partial^2 \boldsymbol{p}^s}{\partial s^\alpha \partial s^\beta} - \frac{\partial \boldsymbol{p}^s}{\partial s^\alpha} \cdot \frac{\partial \boldsymbol{p}^s}{\partial s^\beta}$$

where obviously s^α in the above expressions depend on $\boldsymbol{p}^-$.

Assuming, for reason of consistency that will become apparent subsequently, that $k' \equiv Ek$ and rescaling the shell thickness coordinate $\theta^3 \equiv E\Xi$, the minimal value for the bifurcation functional is found from the following variational statement.

$$\int_A \Big\{ \int_{-Z}^{Z} \Big[\bar{L}^{i3k3} \Big(\frac{1}{E} \frac{\partial \bar{v}_k}{\partial \Xi} \Big) \Big(\frac{1}{E} \frac{\partial \bar{v}_i}{\partial \Xi} \Big) + (\bar{L}^{i\gamma k3}\, Q^{\alpha}_{\gamma}) \Big(\frac{1}{E} \frac{\partial \bar{v}_k}{\partial \Xi} \Big) \Big(\frac{\partial \delta \bar{v}_i}{\partial \theta^\alpha} - T^{r}_{i\alpha}\, \delta \bar{v}_r \Big) + (\bar{L}^{i3k\delta} Q^{\beta}_{\delta}) \Big(\frac{\partial \bar{v}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \bar{v}_s \Big) \Big(\frac{1}{E} \frac{\partial \delta \bar{v}_i}{\partial \Xi} \Big) + (\bar{L}^{i\gamma k\delta} Q^{\alpha}_{\gamma} Q^{\beta}_{\delta}) \Big(\frac{\partial \bar{v}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \bar{v}_s \Big) \Big(\frac{\partial \delta \bar{v}_i}{\partial \theta^\alpha} - T^{r}_{i\alpha} \delta \bar{v}_r \Big) \Big] M d\Xi + k [\bar{D}^{ij} \bar{v}_j \delta \bar{v}_i M]_{\Xi=-Z} \Big\} dA = B \int_A \Big\{ \int_{-Z}^{Z} [A^{ij} \bar{v}_j \delta \bar{v}_i] M d\Xi \Big\} dA \qquad (21)$$

where $M \equiv det M^{i}_{j} = det M^{\alpha}_{\beta}$ and A denotes the domain of definition for the midsurface coordinate θ^α. The calculation of the thickness expansions for the minimum eigenvalue and the corresponding mode proceed by grouping all the terms of Eq. (21) of the like order in E. For this we need the following power series expansions of the minimum eigenvalue B, the bifurcation mode $\bar{v}$, the critical load λ_{cr} and other variables in terms of E.

$$B(\lambda, E) = \overset{0}{B} + \overset{1}{B} E + \overset{2}{B} E^2 + \cdots$$
$$\bar{v}_i(\lambda; \theta^j, z) = \overset{0}{\bar{v}}_i(\lambda; \theta^\alpha, \Xi) + \overset{1}{\bar{v}}_i(\lambda; \theta^\alpha, \Xi) E + \overset{2}{\bar{v}}_i(\lambda; \theta^\alpha, \Xi) E^2 + \cdots$$
$$\lambda_{cr}(E) = \lambda_0 + \lambda_1 E + \lambda_2 E^2 + \cdots$$
$$\bar{L}^{ijkl}(\lambda; \theta^i, E) = \overset{0}{\bar{L}}{}^{ijkl}(\lambda; \theta^\alpha, \Xi) + \overset{1}{\bar{L}}{}^{ijkl}(\lambda; \theta^\alpha, \Xi) E + \overset{2}{\bar{L}}{}^{ijkl}(\lambda; \theta^\alpha, \Xi) E^2 + \cdots$$
$$M^{i}_{j}(\theta^k, E) = \overset{0}{M}{}^{i}_{j} + \overset{1}{M}{}^{i}_{j} E \; (\overset{0}{M}{}^{i}_{j} = \delta^{i}_{j}, \overset{1}{M}{}^{\alpha}_{\beta} = -\Xi B^{\alpha}_{\alpha}, \overset{1}{M}{}^{3}_{j} = \overset{1}{M}{}^{i}_{3} = 0) \qquad (22)$$
$$Q^{i}_{j}(\theta^k, E) = \overset{0}{Q}{}^{i}_{j} + \overset{1}{Q}{}^{i}_{j} E + \overset{2}{Q}{}^{i}_{j} E^2 + \cdots (\overset{0}{Q}{}^{i}_{j} = \delta^{i}_{j})$$
$$M(\theta^k, E) = \overset{0}{M} + \overset{1}{M} E + \overset{2}{M} E^2 \; (\overset{0}{M} = 1, \overset{1}{M} = -\Xi B^{\alpha}_{\alpha}, \overset{2}{M} = \Xi^2 det(B^{\alpha}_{\beta}))$$
$$\bar{D}^{ij} = \overset{0}{\bar{D}}{}^{ij} + \overset{1}{\bar{D}}{}^{ij} E + \overset{2}{\bar{D}}{}^{ij} E^2 + \cdots$$

where $\overset{0}{B}$, $\overset{1}{B}$, $\overset{2}{B}$, etc. are defined as the lowest order term, the first order term, the second order term, etc., respectively in power series expansion of B in terms of E, i. e., $\overset{0}{B} \equiv \lim_{E \to 0} B(E)$, and in the same manner other variables are defined.

The first two terms in the expansion of Eq. (21), i. e., the $O(E^{-2})$ and $O(E^{-1})$ terms will be the following equations Eq. (23) and Eq. (25) respectively from which one deduces Eq. (24) and Eq. (26)

$$O(E^{-2}) : \int_A \Big\{ \int_{-Z}^{Z} \Big[\overset{0}{\bar{L}}{}^{ijkl} \frac{\partial \overset{0}{\bar{v}}_k}{\partial \Xi} \frac{\partial \delta \bar{v}_k}{\partial \Xi} \Big] d\Xi \Big\} dA = 0 \qquad (23)$$

$$\frac{\partial \overset{0}{\bar{v}}_k}{\partial \Xi} = 0 \leftrightarrow \overset{0}{\bar{v}}_k = \overset{0}{\bar{v}}_k(\lambda, \theta^\alpha) \qquad (24)$$

$$O(E^{-1}) : \int_A \Big\{ \int_{-Z}^{Z} \Big[\overset{0}{\bar{L}}{}^{i3k3} \frac{\partial \overset{1}{\bar{v}}_k}{\partial \Xi} + \overset{0}{\bar{L}}{}^{i3k\beta} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big) \Big] \frac{\partial \delta \bar{v}_i}{\partial \Xi} d\Xi \Big\} dA = 0 \qquad (25)$$

$$\overset{0}{\bar{L}}{}^{i3j3} \frac{\partial \overset{1}{\bar{v}}_k}{\partial \Xi} + \overset{0}{\bar{L}}{}^{i3j\beta} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{\beta}_{k\beta} \overset{0}{\bar{v}}_s \Big) = 0 \qquad (26)$$

The O(1) term in Eq. (21) gives

$$O(1) : \int_A \Big\{ \int_{-Z}^{Z} \Big[\Big[\overset{0}{\bar{L}}{}^{i3k3} \frac{\partial \overset{2}{\bar{v}}_k}{\partial \Xi} + (\bar{L}^{i3k3} M)^{1} \frac{\partial \overset{1}{\bar{v}}_k}{\partial \Xi} + \overset{0}{\bar{L}}{}^{i3k\beta} \Big(\frac{\partial \overset{1}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{1}{\bar{v}}_s \Big) + (\bar{L}^{i3k\delta} Q^{\beta}_{\delta} M)^{1} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big) \Big] \frac{\partial \delta \bar{v}_i}{\partial \Xi} + \Big[\overset{0}{\bar{L}}{}^{i\alpha k3} \frac{\partial \overset{1}{\bar{v}}_k}{\partial \Xi} + \overset{0}{\bar{L}}{}^{i\alpha k\beta} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big) \Big] \Big(\frac{\delta \bar{v}_i}{\partial \theta^\alpha} - T^{r}_{i\alpha} \delta \bar{v}_r \Big) - \overset{0}{B} A^{ij} \overset{0}{\bar{v}}_j \, \delta \bar{v}_i \Big] d\Xi + k \Big[\overset{0}{\bar{D}}{}^{ij} \overset{0}{\bar{v}}_j \, \delta \bar{v}_i \Big]_{\Xi=-Z} \Big\} dA = 0 \qquad (27)$$

Once again the simplifying assumptions of the slow variation of Z with respect to θ^α and of the Ξ independence of $\overset{0}{\bar{L}}$ will be used in the subsequent analysis. The Euler equation of Eq. (27) and the corresponding boundary condition are found to be

$$\frac{\partial}{\partial \Xi}[\alpha^i] + \frac{\partial}{\partial \theta^\alpha} \Big[\overset{0}{\bar{P}}{}^{i\alpha k\beta} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big) \Big] + (\overset{0}{\bar{P}}{}^{j\alpha k\beta} T^{i}_{j\alpha} + \overset{0}{\bar{P}}{}^{i\alpha k\beta} T^{\gamma}_{\alpha\gamma}) \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big) + \overset{0}{B} A^{ij} \overset{0}{\bar{v}}_j = 0 \; ; \; \Xi \in [-Z, Z]$$
$$\alpha_i = 0 \; at \; \Xi = Z, \; \alpha^i = k \, \overset{0}{\bar{D}}_{ij} \overset{0}{\bar{v}}_j \; at \; \Xi = -Z \qquad (28)$$
$$\alpha^i \equiv \overset{0}{\bar{L}}{}^{i3k3} \frac{\partial \overset{2}{\bar{v}}_k}{\partial \Xi} + \overset{1}{\bar{L}}{}^{i3k3} \frac{\partial \overset{1}{\bar{v}}_k}{\partial \Xi} + \overset{0}{\bar{L}}{}^{i3k\beta} \Big(\frac{\partial \overset{1}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{1}{\bar{v}}_s \Big) + (\bar{L}^{i3k\delta} Q^{\beta}_{\delta})^{1} \Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta} \overset{0}{\bar{v}}_s \Big),$$
$$\bar{P}^{ijkl} \equiv \bar{L}^{irk\delta} Q^{j}_{r} Q^{l}_{\delta} - \bar{L}^{irp3} Q^{j}_{r} (\bar{L}^{m3p3})^{-1} \bar{L}^{m3k\delta} Q^{l}_{\delta} \equiv \overset{0}{\bar{P}}{}^{ijkl} + \overset{1}{\bar{P}}{}^{ijkl} E + \overset{2}{\bar{P}}{}^{ijkl} E^2 + \cdots$$

with the usual notation of **P** for the plane stress incremental moduli and definitions same as in Eq. (22). Integration of Eq. (28) in the interval $[-Z, Z]$, or equivalently consideration of Eq. (27) for δv independent of Ξ results in

$$\alpha^i = \Big(\frac{Z - \Xi}{Z} \Big) \frac{k}{2} \overset{0}{\bar{D}}{}^{ij} \overset{0}{\bar{v}}_j$$

$$\int_A \Big\{ 2Z\Big[\overset{0}{P}{}^{i\alpha k\beta}\Big(\frac{\partial \overset{0}{\bar v}_k}{\partial\theta^\beta} - T^s_{k\beta}\,\overset{0}{\bar v}_s\Big)\Big(\frac{\partial\delta\bar v_i}{\partial\theta^\beta} - T^r_{i\alpha}\delta\bar v_r\Big) - \overset{0}{B}A^{ij}\,\overset{0}{\bar v}_j\,\delta\bar v_i\Big] + k\,\overset{0}{\bar D}{}^{ij}\,\overset{0}{\bar v}_j\,\delta\bar v_i\Big\} dA = 0 \Big(\frac{\partial\delta\bar v_i}{\partial\Xi}=0\Big) \quad (29)$$

The calculation of $\overset{0}{B}$ from Eq. (29) is a matter of a straightforward substitution of $\delta v=\overset{0}{v}$ in Eq. (29) and hence $\overset{0}{B}$ is found to be

$$\overset{0}{B}=\int_A\Big\{\overset{0}{P}{}^{i\alpha k\beta}\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\,\overset{0}{\bar v}_s\Big)\Big(\frac{\partial\overset{0}{\bar v}_i}{\partial\theta^\alpha}-T^r_{i\alpha}\,\overset{0}{\bar v}_r\Big) + \frac{k}{2Z}\,\overset{0}{\bar D}{}^{ij}\,\overset{0}{\bar v}_j\,\overset{0}{\bar v}_i\Big\}dA\Big/\int_A\Big\{A^{ij}\,\overset{0}{\bar v}_j\,\overset{0}{\bar v}_i\Big\}dA \quad (30)$$

The next term in the thickness expansion of Eq. (21) yields

$$\begin{aligned}O(E):\ \int_A\Big\{\int_{-Z}^{Z}\Big[\Big[&\overset{0}{\bar L}{}^{i3k3}\frac{\partial\overset{3}{\bar v}_k}{\partial\Xi}+(\bar L^{i3k3}M)^1\frac{\partial\overset{2}{\bar v}_k}{\partial\Xi} + (\bar L^{i3k3}M)^2\frac{\partial\overset{1}{\bar v}_k}{\partial\Xi}+\overset{0}{\bar L}{}^{i3k\beta}\Big(\frac{\partial\overset{2}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\overset{2}{\bar v}_s\Big)\\ &+(\bar L^{i3k\delta}Q^\beta_\delta M)^1\Big(\frac{\partial\overset{1}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\overset{1}{\bar v}_s\Big)+(\bar L^{i3k\delta}Q^\beta_\delta M)^2\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta} - T^s_{k\beta}\overset{0}{\bar v}_s\Big)\Big]\frac{\partial\delta\bar v_i}{\partial\Xi}+\Big[\overset{0}{\bar L}{}^{i\alpha k3}\frac{\partial\overset{2}{\bar v}_k}{\partial\Xi}+(\bar L^{i\gamma k3}Q^\alpha_\gamma M)^1\frac{\partial\overset{1}{\bar v}_k}{\partial\Xi}\\ &+\overset{0}{\bar L}{}^{i\alpha k\beta}\Big(\frac{\partial\overset{1}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\overset{1}{\bar v}_s\Big)+(\bar L^{i\gamma k\delta}Q^\alpha_\gamma Q^\beta_\delta M)^1\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta} - T^s_{k\beta}\overset{0}{\bar v}_s\Big)\Big]\Big(\frac{\partial\delta\bar v_i}{\partial\theta^\alpha}-T^r_{i\alpha}\delta\bar v_r\Big)-A^{ij}\Big[\overset{0}{B}\overset{1}{\bar v}_j\\ &+(BM)^1\overset{0}{\bar v}_j\Big]\delta\bar v_i\Big]d\Xi+k\Big[\Big[\overset{0}{\bar D}{}^{ij}\overset{1}{\bar v}_j + (\bar D^{ij}M)^1\,\overset{0}{\bar v}_j\Big]\delta\bar v_i\Big]_{\Xi=-Z}\Big\}dA=0\end{aligned} \quad (31)$$

The above equation for $\delta v=\delta v(\theta^\alpha,\Xi)$ provides an equation for $\dfrac{\partial\overset{3}{\bar v}_k}{\partial\Xi}$, while for $\delta v=\delta v(\theta^\alpha)$ one obtains the following variational equation for $\langle\overset{1}{\bar v}_k\rangle$ (the thickness average of $\overset{1}{\bar v}_k$)

$$\begin{aligned}\int_A\Big\{2Z\Big[\Big[&\overset{0}{P}{}^{i\alpha k\beta}\Big(\frac{\partial\langle\overset{1}{\bar v}_k\rangle}{\partial\theta^\beta}-T^s_{k\beta}\langle\overset{1}{\bar v}_s\rangle\Big) +\langle\overset{1}{P}{}^{i\alpha k\beta}\rangle\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\,\overset{0}{\bar v}_s\Big)\\ &+\frac{k}{2}\,\overset{0}{\bar L}{}^{i\alpha k3}(\overset{0}{\bar L}{}^{l3k3})^{-1}\,\overset{0}{\bar D}{}^{lj}\,\overset{0}{\bar v}_j\Big]\Big(\frac{\partial\delta\bar v_i}{\partial\theta^\alpha}-T^r_{i\alpha}\,\delta\bar v_r\Big) -\Big[\overset{0}{B}A^{ij}\langle\overset{1}{\bar v}_j\rangle+\overset{1}{B}A^{ij}\,\overset{0}{\bar v}_j\Big]\delta\bar v_i\Big]+k\Big[\overset{0}{\bar D}{}^{ij}\big(\langle\overset{1}{\bar v}_j\rangle\\ &+Z(\overset{0}{\bar L}{}^{l3j3})^{-1}\overset{0}{\bar L}{}^{l3k\beta}\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\,\overset{0}{\bar v}_s\Big)\big) +(\bar D^{ij}M)^1{}_{\Xi=-Z}\,\overset{0}{\bar v}_j\Big]\delta\bar v_i\Big\}dA=0\ \Big(\frac{\partial\delta\bar v_i}{\partial\Xi}=0\Big)\end{aligned} \quad (32)$$

The term $\overset{1}{B}$ in the expansion of the minimum eigenvalue of the bifurcation functional is obtained from Eq. (32) by taking $\delta v=\overset{0}{v}$ and by recalling Eq. (29). Hence

$$\begin{aligned}\overset{1}{B}=\int_A\Big\{&\langle\overset{1}{P}{}^{i\alpha k\beta}\rangle\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\overset{0}{\bar v}_s\Big)\Big(\frac{\partial\overset{0}{\bar v}_i}{\partial\theta^\alpha}-T^r_{i\alpha}\overset{0}{\bar v}_r\Big) +k\overset{0}{\bar L}{}^{i\alpha k3}(\overset{0}{\bar L}{}^{l3k3})^{-1}\overset{0}{\bar D}{}^{lj}\overset{0}{\bar v}_j\Big(\frac{\partial\overset{0}{\bar v}_i}{\partial\theta^\alpha}-T^r_{i\alpha}\overset{0}{\bar v}_r\Big)\\ &+\frac{k}{2Z}(\bar D^{ij}M)_{\Xi=-Z}\overset{0}{\bar v}_i\overset{0}{\bar v}_j\Big\}dA\Big/\int_A\{A^{ij}\overset{0}{\bar v}_i\overset{0}{\bar v}_j\}dA\end{aligned} \quad (33)$$

The above results will be employed in the calculation of the minimum value of the bifurcation functional F within an error of $O(E^4)$ for a given value of the load parameter λ, a result which as will be discussed later is extremely valuable in numerical solutions to sheet metal forming surface instability problems. Indeed, one has

$$\begin{aligned}\mathrm{F}=\frac{E}{2}\int_A\Big\{\int_{-Z}^{Z}\Big[&\bar L^{i3k3}\Big(\frac{1}{E}\frac{\partial\bar v_i}{\partial\Xi}\Big)\Big(\frac{1}{E}\frac{\partial\bar v_k}{\partial\Xi}\Big) +(\bar L^{i\gamma k3}Q^\alpha_\gamma)\Big(\frac{1}{E}\frac{\partial\bar v_k}{\partial\Xi}\Big)\Big(\frac{\partial\bar v_i}{\partial\theta^\alpha}-T^r_{i\alpha}\bar v_r\Big)\\ &+(\bar L^{i3k\delta}Q^\beta_\delta)\Big(\frac{\partial\bar v_k}{\partial\theta^\beta}-T^s_{k\beta}\bar v_s\Big)\Big(\frac{1}{E}\frac{\partial\bar v_i}{\partial\Xi}\Big) +(\bar L^{i\gamma k\delta}Q^\alpha_\gamma Q^\beta_\delta)\Big(\frac{\partial\bar v_k}{\partial\theta^\beta}-T^s_{k\beta}\bar v_s\Big)\Big(\frac{\partial\bar v_i}{\partial\theta^\alpha}\\ &-T^r_{i\alpha}\bar v_r\Big)\Big]Md\Xi+k[\bar D^{ij}\bar v_i\bar v_jM]_{\Xi=-Z}\Big\}dA\end{aligned} \quad (34)$$

Substitution of the asymptotic expansions in Eq. (22), Eq. (23), Eq. (25) and Eq. (29) into Eq. (34) and keeping all the terms up to $O(E^3)$ (inclusive), noting the mode orthogonality, yields the following expression for the bifurcation functional F (within an $O(E^4)$ error)

$$\begin{aligned}\mathrm{F}=\frac{E}{2}\int_A\Big\{\int_{-Z}^{Z}\Big[&(\overset{0}{P}{}^{i\alpha k\beta}+E\overset{1}{P}{}^{i\alpha k\beta} +E^2\overset{2}{P}{}^{i\alpha k\beta})\Big(\frac{\partial(\overset{0}{\bar v}_k+E\ \overset{1}{\bar v}_k)}{\partial\theta^\beta} -T^s_{k\beta}(\overset{0}{\bar v}_s+E\overset{1}{\bar v}_s)\Big)\frac{\partial(\overset{0}{\bar v}_i+E\ \overset{1}{\bar v}_i)}{\partial\theta^\alpha}\\ &-T^r_{i\alpha}(\overset{0}{\bar v}_r+E\overset{1}{\bar v}_r))\cdot(\overset{0}{M}+E\overset{1}{M}+E^2\overset{2}{M})\Big]d\Xi\\ &+k\Big[(\overset{0}{\bar D}{}^{ij}+E\overset{1}{\bar D}{}^{ij}+E^2\overset{2}{\bar D}{}^{ij})(\overset{0}{\bar v}_j+E\overset{1}{\bar v}_j)(\overset{0}{\bar v}_i+E\overset{1}{\bar v}_i) (\overset{0}{M}+E\overset{1}{M}+E^2\overset{2}{M})]_{\Xi=-Z}\\ &+E^2<\int_{-Z}^{Z}\Big[-\frac{k^2}{4}\Big(\frac{Z-\Xi}{Z}\Big)^2\overset{0}{\bar D}{}^{im}(\overset{0}{\bar L}{}^{n3m3})^{-1}\overset{0}{\bar D}{}^{nj}\overset{0}{\bar v}_j\overset{0}{\bar v}_i\\ &+k\Big(\frac{Z-\Xi}{Z}\Big)\overset{0}{\bar D}{}^{ij}(\overset{0}{\bar L}{}^{m3j3})^{-1}\Big[\overset{0}{\bar L}{}^{m3k\beta}\Big(\frac{\partial\overset{1}{\bar v}_k}{\partial\theta^\beta}-T^s_{k\beta}\overset{1}{\bar v}_s\Big)\\ &+\Big((\bar L^{m3k\delta}Q^\beta_\delta)^1\overset{1}{\bar L}{}^{m3q3}(\overset{0}{\bar L}{}^{p3q3})^{-1}\overset{0}{\bar L}{}^{p3k\beta}\Big)\Big(\frac{\partial\overset{0}{\bar v}_k}{\partial\theta^\beta}\end{aligned}$$

$$- T^{s}_{k\beta} \overset{0}{\bar{v}}_s\Big)\Big]\overset{0}{\bar{v}}_i\Big]d\Xi > \Big\} dA \tag{35}$$

where $\overset{1}{\bar{v}}_i = -\Xi (\overset{0}{L}{}^{m3i3})^{-1} \overset{0}{L}{}^{m3k\beta}\left(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta}\overset{0}{\bar{v}}_s\right) + \langle \overset{1}{\bar{v}}_i \rangle$

Although Eq. (35) is the exact (up to $O(E^3)$) value for the minimum of the bifurcation functional, a slightly simplified version has been employed in applications (Triantafyllidis, 1980 ; Triantafyllidis and Needleman, 1980). This simplification is based on the observation that for adequately high values of k, the E^2 correction under the angular bracket in Eq. (35) is negligible. Indeed, from Eq. (29), $O(kD\mathbf{v}) = O(L\partial\mathbf{v}/\partial\theta) = O(L)$, where the second equation fixes the magnitude of the eigenmode. In the areas of contact between the sheet and the punch the large values of (kD/L) provide locally a eigenmode $\mathbf{v}$ close to zero $O(L/kD)$ and thus any correction to the vanishing eigenmode at this point is negligible (recall that elsewhere $\mathbf{v} = O(1)$).

An additional convenient approximation (justified by the fact $\langle \overset{1}{P}{}^{i\alpha k\beta} \rangle \cong 0$ in applications — see also Eq. (32)) is omission of $\langle \overset{1}{\bar{v}}_k \rangle$ and hence, a simplified version of the bifurcation functional Eq. (35) can be given, namely,

$$\mathrm{F} = \frac{1}{2}\int_A \Big\{ \int_{-H}^{H} \Big[\bar{P}^{i\alpha k\beta}(\lambda\,;\,\theta^i, E)\,\Big(\frac{\partial \hat{v}_i}{\partial \theta^\alpha} - T^{r}_{i\alpha}\,\hat{v}_i\Big)\Big(\frac{\partial \hat{v}_k}{\partial \theta^\beta} - T^{s}_{k\beta}\,\hat{v}_s\Big) M(\theta^i)\Big] d\theta^3 + k\bar{D}^{ij}(\lambda\,;\,\theta^i, E)\;\hat{v}_i\;\hat{v}_j \Big\} dA \tag{36}$$

with $\hat{v}_i(\lambda;\theta^i, E) = \overset{0}{\bar{v}}_i - \theta^3 (\bar{L}^{j3i3}(\lambda;\theta^i, E))^{-1} \bar{L}^{j3k\beta}(\lambda; \theta^i, E)\Big(\frac{\partial \overset{0}{\bar{v}}_k}{\partial \theta^\beta} - T^{s}_{k\beta}\,\overset{0}{\bar{v}}_s\Big)$

Hence, in sheet metal forming applications, the following procedure is used for the determination of the critical load corresponding to the onset of a surface instability.

After computing from the membrane stresses, for a given externally applied load λ (usually λ is the rigid punch advance), one can construct the plane stress incremental moduli (see Eq. (10) and Eq. (12)) and hence investigate the positive definiteness of F in Eq. (35) or Eq. (36). The first time that loses its positive definiteness, as λ increases indicates the onset of an instability.

The aforeproposed methodology for the calculation of the onset of a surface waviness instability in a shell has the obvious advantage over the classical approach of not requiring a nonlinear shell theory while using only the membrane stress state in the sheet, a result which can be routinely (and uniquely for a given sheet geometry and material) calculated using existing sheet metal forming codes.

4. Conclusions

In the present work a consistent and general approach for the analysis of the onset of buckling instabilities in shell like structures of arbitrary geometry and material properties is proposed. In place of the classical approach of linearizing the governing nonlinear shell equations about the critical load, in the present method the full three dimensional linearized bifurcation equations for the solid are asymptotically expanded with respect to a small parameter.

The natural selection for the small parameter is the maximum (or minimum) reference configuration shell thickness. In the limiting process considered, the reference middle surface remains fixed as the shell's thickness tends to zero, thus making the limiting process depending only on one small parameter (unlike the work on linearly elastic shells by Destuynder(1980) where two small parameters were needed). The method results in the solution of a sequence of two dimensional boundary value problems whose domain is the reference middle surface of the shell.

The theory is valid for shells with arbitrary geometry and contact loading with a rigid punch, with the case being particularly useful to metal forming applications. The constitutive law for the shell can be any rate independent model, thus including in the analysis all hyperelastic, hypoelastic or elastoplastic materials.

As explained in the introduction, the proposed methodology presents some theoretical as well as some practical advantages over the classical method. The theoretical advantages are the non-dependence (for a given structure) of the critical load and mode on the nonlinear shell theory employed and the possibility in obtaining higher order terms if necessary for the thickness dependence of these quantities. The practical advantages are especially useful in the case of modelling the onset of surface waviness instabilities in sheet metal forming. The first advantage is the absence of necessity for a nonlinear shell theory in modelling this process, which in addition of being costlier than a membrane model presents the still unsolved problem of its proper selection. The second advantage is that only the plane stress moduli and the membrane stresses are required in the determination (with the minimum necessary accuracy) of the onset of a buckling instability, quantities which are already available from existing membrane codes modelling sheet forming processes.

Of course, since the issue of shell buckling is a fairly old and difficult one, the present approach due to its novelty leaves a number of questions still open. The questions that have to be answered next are more technical and complex. One concerns the method's accuracy. Some interesting preliminary investigations in this direction were presented by Triantafyllidis and Kwon (1987) for a pressurized infinite cylinder with very encouraging results. Due to the highly nonlinear nature of the problem some more work in this direction is necessary. On the experimental side some cup puckering tests were conducted (Donoghue et al., 1989) and compared with the analysis presented in section 3. The test results were in remarkably good agreement with the theoretical predictions for this type of experiments. Another concern is about boundary layer phenomena which are typical to multiple scales asymptotic expansions methods (Bensoussan et al., 1978).

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